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# Anomalous Diffusion Index for Lévy Motions

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In modelling complex systems as real diffusion processes it is common to analyse its diffusive regime through the study of approximating sequences of random walks. For the partial sums $S_n = \xi_1 + \xi_2 + \ldots + \xi_n$ one considers the approximating sequence of processes $X^{(n)}(t) = a_n(S_{[k_n t]} - b_n)$. Then, under sufficient smoothness requirements we have the convergence to the desired diffusion, $X^{(n)}(t) \to X(t)$. A key assumption usually presumed is the finiteness of the second moment, and, hence the validity of the Central Limit Theorem. Under anomalous diffusive regime the asymptotic behavior of $S_n$ may well be non-Gaussian and $n^{-1}E(S_n^2) \to \infty$. Such random walks have been referred by physicists as Lévy motions or Lévy flights. In this work, we introduce an alternative notion to classify these regimes, the diffusion index $\gamma_X$. For some $\gamma_X^0$ properly chosen let $\gamma_X = \inf\{\gamma : 0 < \gamma \leq \gamma_X^0, \limsup_{t\to\infty} t^{-1}E|X(t)|^{1/\gamma} < \infty\}$. Relationship between $\gamma_X$, the infinitesimal diffusion coefficients and the diffusion constant will be explored. Illustrative examples as well as estimates, based on extreme order statistics, for $\gamma_X$ will also be presented.

**KEY WORDS:** Lévy motions, anomalous diffusion.

## 1. INTRODUCTION

Random motions of particles in space are generally modelled as diffusions. Classical or normal diffusion occurs when the mean square displacement during a time interval becomes, for sufficient long intervals, a linear function of it. From a mathematical point of view, this phenomenon is a consequence of the Central Limit Theorem which is concerned with the statistical properties of the cumulative displacements arising from a very large number of independent displacements. By taking into account the independence of the increments, in classical Probability Theory, a diffusion process can be viewed as a path continuous Markov process

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$\{X(t) : t \geq 0\}$ with local diffusion characterized by its infinitesimal parameters: for every $x, t > 0$ and $\epsilon > 0$,

$$\lim_{h \downarrow 0} \frac{1}{h} P\left(|X(t+h) - X(t)| > \epsilon \,| X(t) = x\right) = 0,$$

$$\lim_{h \downarrow 0} \frac{1}{h} E\left\{X(t+h) - X(t) \,| X(t) = x\right\} = \mu_X(x, t), \tag{1}$$

$$\lim_{h \downarrow 0} \frac{1}{h} E\left\{(X(t+h) - X(t))^2 \,| X(t) = x\right\} = \sigma_X^2(x, t).$$

The functions $\mu_X(\cdot)$ and $\sigma_X^2(\cdot)$ are termed, respectively, drift coefficient (infinitesimal mean) and diffusion coefficient (infinitesimal variance). The basic and simplest example is the Brownian motion $\{B(t) : t \geq 0\}$ where $\mu_B(x, t) = 0$ and $\sigma^2(x, t) = 1$.

On the other hand, in applied stochastic modelling it is common to analyse diffusion processes through the study of approximating sequences of random walks. One considers $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ where $\xi_1, \xi_2, \ldots$ are i.i.d. (independent and identically distributed) random variables and defines an approximating sequence of processes

$$X^{(n)}(t) = a_n \,(S_{[k_n t]} - b_n).$$

Where $b_n$ is a centering constant, $a_n$ performs a scaling of the state variable, $k_n$ performs the required time scaling and $[k_n t]$ denotes the largest integer not exceeding $k_n t$. Then, under sufficient smoothness requirements we have the approximation to the desired diffusion, $X^{(n)}(t) \to X(t)$. For example, the Donsker's Functional Central Limit Theorem assures that if $\xi_j$'s have zero mean and finite variance $E(\xi_j^2) = \sigma^2 > 0$ then

$$X^{(n)}(t) = \frac{1}{\sigma\sqrt{n}} \{S_{[nt]} + (nt - [nt])\xi_{[nt]+1}\} \to B(t). \tag{2}$$

A key assumption necessary for the convergence (2) is the finiteness of the second moment, and, hence the validity of the Central Limit Theorem. In which case one should expect that $0 < \lim_{t \to \infty} \frac{E(X^2(t))}{2t} < \infty$, leading to a normal diffusive regime. In anomalous diffusion, this linearity breaks down to yields to a form of diffusion either faster or slower than normal. Such anomalous diffusion and their corresponding approximating random walks have been referred by physicists as Lévy motions or Lévy flights. Their applicability have appeared in geophysics, hydrology, turbulence, economics and physics, particularly, in connection to the so-called renormalization of group theory (RGT) and critical phenomena or anomalous diffusions. This led physicists to make use of the following limit to classify distinct diffuse regimes :

$$D_X = \lim_{t\to\infty} \frac{E(X^2(t))}{2t}. \tag{3}$$

For such concepts and applications see, for example, [4,5,7,8–9] and references there in. Since for the Brownian motion with variance $\sigma^2$ the diffusion constant is $D_B = \sigma^2/2$, the following classification is used:

$$\begin{aligned} D_X = 0 \quad &\Rightarrow \quad \text{subdiffusion} \\ 0 < D_X < \infty \quad &\Rightarrow \quad \text{normal diffusion} \\ D_X = \infty \quad &\Rightarrow \quad \text{superdiffusion.} \end{aligned} \tag{4}$$

In this work, as in the diffusion constant $D_X$, we analyse the long-range diffusion behavior. In Sec 2 we introduce a notion of diffusion index $\gamma_X$, based on the analysis of

$$\liminf_{t\to\infty} \frac{E(|X(t)|^{1/\gamma})}{t} \quad \text{and} \quad \limsup_{t\to\infty} \frac{E(|X(t)|^{1/\gamma})}{t}$$

where $\gamma > 0$. One shows, among other results, that $\gamma_X = \gamma_*$, if for some $\gamma_* > 0$ we have $0 < \lim_{t\to\infty} t^{-1}E(|X(t)|^{1/\gamma_*} < \infty$. Relationship between $\gamma_X$, the infinitesimal coefficients and the diffusion constant will also be explored.

Our Proposition 5 shows that under superdiffusive regime and for Lévy processes the index $\gamma_X$ constitute a refinement of the diffusion constant $D_X$. Since stable Lévy distributions are limits in law of stabilized sums $\frac{S_n - b_n}{n^{\gamma_X}}$, such refinement along with the knowledge of the index $\gamma_X$ will allow us to study anomalous diffusions as limits of properly scaled transformations of random walks. And this constitutes more accurate approximations than the ordinary scaled random walks that are generally used. [3,11,13] By making elementary operations on the process the results from Proposition 2 suggest possible procedures to analyse decomposition, superposition or relaxation of diffusions.

In Sec 4 we further explore the role played by stable distributions in diffusion analysis. It will be seen that $\frac{1}{n}E\left(S_n^2\right) \to \infty$ along with its tail behavior will lead us to an estimate of the index $\gamma_X$ . Some illustrative examples are included in Sec 3, particularly, Example 2 shows that $D_X$ does not avoid misleading diffusions caused by the drift and that this can be bypassed by considering centered processes.

## 2. DIFFUSION INDEX

Assume $\{X(t) : t \geq 0\}$ is a real centered stochastic process, that is, a real valuated process with either $E\{X(t)\} = 0$ or satisfying the symmetry

$$P(X(t) \leq x) = P(X(t) \geq -x), \quad \forall x, \quad \forall t.$$

Instead of considering local diffusion characteristics as in (1) we explore the long-range properties as in (3).

**Definition 1.** For a centered stochastic process $X(\cdot)$ let

$$\gamma_X^0 = \sup\{\gamma : \gamma > 0, \ \liminf_{t\to\infty} \frac{E|X(t)|^{1/\gamma}}{t} > 0\}, \tag{5}$$

$$\gamma_X^0 = 0 \quad \text{if} \quad \liminf_{t\to\infty} \frac{E|X(t)|^{1/\gamma}}{t} = 0, \quad \forall \gamma > 0.$$

And define the diffusion index $\gamma_X$ by

$$\gamma_X = \inf\{\gamma : 0 < \gamma \leq \gamma_X^0, \ \limsup_{t\to\infty} \frac{E|X(t)|^{1/\gamma}}{t} < \infty\} \tag{6}$$

$$\gamma_X = 0 \quad \text{if} \quad \gamma_X^0 = 0,$$

$$\gamma_X = \infty \quad \text{if} \quad \liminf_{t\to\infty} \frac{E|X(t)|^{1/\gamma}}{t} = \infty, \quad \forall \gamma > 0.$$

**Proposition 1.** If $0 < \gamma_X < \infty$ then $\gamma_X = \gamma_X^0$,

$$\limsup_{t\to\infty} \frac{E|X(t)|^{1/\gamma}}{t} = \infty \quad \text{for} \quad 0 < \gamma < \gamma_X \tag{7}$$

and

$$\liminf_{t\to\infty} \frac{E|X(t)|^{1/\gamma}}{t} = 0 \quad \text{for} \quad \gamma > \gamma_X. \tag{8}$$

**Proof.** Directly from (6) we obtain (7). To prove (8) we make use of Liapounov's inequality : for $\gamma > \gamma_1 > 0$

$$(E|X(t)|^{1/\gamma})^{\gamma} \leq (E|X(t)|^{1/\gamma_1})^{\gamma_1}$$

and

$$t^{\gamma-\gamma_1}\left(\frac{E|X(t)|^{1/\gamma}}{t}\right)^{\gamma} \leq \left(\frac{E|X(t)|^{1/\gamma_1}}{t}\right)^{\gamma_1} \tag{9}$$

If for some $\gamma > \gamma_X$, we have $\liminf_{t\to\infty} \frac{E|X(t)|^{1/\gamma}}{t} > 0$ then, by (9), we have

$$\limsup_{t\to\infty} \frac{E|X(t)|^{1/\gamma_1}}{t} = \infty \quad , \quad 0 < \gamma_1 < \gamma.$$

From (6) we have $\gamma_X \geq \gamma$, a contradiction. Thus (8) follows and by (5) we have $\gamma_X^0 \leq \gamma_X$. Using again (6) we have $\gamma_X = \gamma_X^0$. □

**Corollary 1.** Suppose for some $0 < \gamma_* < \infty$ we have

$$0 < \liminf_{t\to\infty} \frac{E|X(t)|^{1/\gamma_*}}{t} \leq \limsup_{t\to\infty} \frac{E|X(t)|^{1/\gamma_*}}{t} < \infty. \tag{10}$$

Then $\gamma_X = \gamma_*$.

This allows an alternative definition for the diffusion index.

**Definition 2.** If (10) is satisfied with $0 < \gamma_* < \infty$ define $\gamma_X = \gamma_*$ . If no such $\gamma_*$ exists define

$$\gamma_X = 0 \quad \text{if} \quad \liminf_{t\to\infty} \frac{E|X(t)|^{1/\gamma}}{t} = 0 \quad , \quad \forall \gamma > 0 \tag{11}$$

and

$$\gamma_X = \infty \quad \text{if} \quad \limsup_{t\to\infty} \frac{E|X(t)|^{1/\gamma}}{t} = \infty \quad , \quad \forall \gamma > 0. \tag{12}$$

**Proposition 2.** Let $X(\cdot)$ and $Y(\cdot)$ be centered stochastic processes. If $a \neq 0$ and $b > 0$ are constants, then

$$\gamma_{|X|} = \gamma_X \quad , \quad \gamma_{aX} = \gamma_X \quad , \quad \gamma_{|X|^b} = b\gamma_X \quad \text{and} \quad \gamma_{X+Y} \leq \max\{\gamma_X, \gamma_Y\}.$$

**Proof.** The equalities are immediate. For the inequality consider the following refinement of Minkowski inequality,

$$E|X+Y|^r \leq C_r\left(E|X|^r + E|Y|^r\right), \quad r \geq 0, \tag{13}$$

where $C_r = 1$ if $0 \leq r \leq 1$ and $C_r = 2^{r-1}$ if $r > 1$.

Clearly, if $\max\{\gamma_X, \gamma_Y\} = 0$ then by (11) and (13) we have $\gamma^0_{X+Y} = \gamma_{X+Y} = 0$. Assume $\gamma > \max\{\gamma_X, \gamma_Y\} > 0$. Then by Proposition 1

$$\liminf_{t\to\infty} \frac{E|X(t)|^{1/\gamma}}{t} = \liminf_{t\to\infty} \frac{E|Y(t)|^{1/\gamma}}{t} = 0.$$

From (13) we conclude that

$$\liminf_{t\to\infty} \frac{E|X(t) + Y(t)|^{1/\gamma}}{t} = 0 \quad , \quad \gamma > \max\{\gamma_X, \gamma_Y\}.$$

From (5) we have $\gamma^0_{X+Y} \le \max\{\gamma_X, \gamma_Y\}$ and from (6) we have $\gamma_{X+Y} \le \max\{\gamma_X, \gamma_Y\}$. □

Next, we compare the index $\gamma_X$ with the diffusion coefficient $\sigma^2_X(x, t)$ and the diffusion constant $D_X$. We say that a process $X(\cdot)$ has stationary and independent increments if

$$(X(t + h)) - X(t)) \stackrel{D}{=} (X(s + h) - X(s)),$$

and for $0 \le t_0 < t_1 < \cdots < t_k$ the random variables $X(t_1) - X(t_0), \ldots, X(t_k) - X(t_{k-1})$ are independent ($\stackrel{D}{=}$ stands for same distribution).

**Proposition 3.** Let $X(\cdot)$ be a zero-mean stochastic process with stationary and independent increments. Then, if $0 < \sigma^2_X(x, t) < \infty$ we have $\gamma_X = 1/2$.

**Proof.** By the stationarity and independence of the increments we have

$$E\{(X(t + h) - X(t))^2 | X(t) = x\} = E(X^2(h))$$

and from (1)

$$\sigma^2_X(x, t) = \lim_{h\downarrow 0} \frac{E\{X^2(h)\}}{h} = a, \quad 0 < a < \infty.$$

Also, for $n \ge 1$ and $h > 0$

$$E(X^2(nh)) = E\left\{\left(\sum_{j=1}^{n} [X(jh) - X((j-1)h)]\right)^2\right\} = nE(X^2(h)).$$

Now let $h_n \downarrow 0$ with $nh_n \to \infty$ then

$$\lim_{t\to\infty} \frac{E(X^2(t))}{t} = \lim_{n\to\infty} \frac{E(X^2(nh_n))}{nh_n} = a.$$

Since $0 < a < \infty$ we have (10) satisfied with $\gamma_* = 1/2$. By Corollary 1 the result follows. □

Similarly, the Liapounov's inequality (10) also gives:

**Proposition 4.** If the diffusion constant $0 < D_X < \infty$ then $\gamma_X = 1/2$.

The role of diffusion index can be better illustrated by the Lévy motions that constitute a subclass of processes with stationary and independent increments. Its increments possess distributions that allow a great variability and that, unlike the Gaussian case, do not possess finite second moments.

We say that $S_{\alpha,\beta}$, $0 < \alpha \leq 2$ and $\beta > 0$, is a symmetric stable distribution with stability index $\alpha$ and scaling parameter $\beta$ if the corresponding characteristic function is given by

$$\phi_{S_{\alpha,\beta}}(\theta) = E(e^{i\theta S_{\alpha,\beta}}) = \exp\{-\beta^\alpha |\theta|^\alpha\}.$$

Note that if $\alpha = 2$ we have the Gaussian case, $N(0, 2\beta^2)$. For a comprehensive survey of properties of stable laws see, for example. [1,12]

**Definition 3.** A stochastic process $L_{\alpha,\sigma} = \{L(t) : t \geq 0\}$ is said to be a symmetric Lévy stable process if :

(i) $L(0) = 0$ a.s. (almost surely).

(ii) $L(\cdot)$ has stationary and independent increments.

(iii) For $0 \leq s < t$, $L(t) - L(s) \stackrel{D}{=} S_{\alpha,\sigma(t-s)^{1/\alpha}}$ for some $0 < \alpha \leq 2$ and $\sigma > 0$.

**Remark 1.** (a) For $\alpha = 2$ we have the Brownian motion with variance $\sigma^2/2$. Since $L(1)$ is Gaussian we have $E|L(1)|^{1/\gamma} < \infty,\ \forall\gamma > 0$ and from (1) and (3) we have $\sigma_L^2(x, t) = \sigma^2$ and $D_L = \sigma^2/2$.

(b) Under scaling of time and space the Lévy stable processes are invariant in distribution. Also, they are self-similar processes satisfying,

$$L(at) - L(as) \stackrel{D}{=} a^{1/\alpha}[L(t) - L(s)]\ ,\ a > 0,\ s \geq 0,\ t \geq 0. \tag{14}$$

(c) For $0 < \alpha < 2$ we have $\alpha = \kappa_L$, the moment index,

$$\kappa_L = \sup\{\kappa : \kappa > 0,\ E|L(1)|^\kappa < \infty\} \tag{15}$$

(see, (1)).

**Proposition 5.** For the Lévy stable process $L = L_{\alpha,\sigma}$, if $\alpha = 2$ then

$$\sigma_L^2(x, t) = \sigma^2, \quad D_L = \frac{\sigma^2}{2} \text{ and } \gamma_L = \frac{1}{2}.$$

And, if $0 < \alpha < 2$ (superdiffusion) we have

$$\sigma_L^2(x,t) = D_L = \infty \quad \text{and} \quad \gamma_L = \frac{1}{\alpha} > \frac{1}{2}.$$

**Proof.** Let $\gamma > 0$ and $t > 0$. Since $L(0) = 0$ we have by (14)

$$\varphi_\gamma(t) = \frac{E|L(t)|^{1/\gamma}}{t} = \frac{t^{1/\alpha\gamma} E|L(1)|^{1/\gamma}}{t}.$$

If $\alpha = 2$ then $E|L(1)|^{1/\gamma} < \infty$ and $\lim_{t\to\infty} \varphi_{1/2}(t) = \sigma^2$. Since $0 < \sigma^2 < \infty$, by Proposition 2, we have $\gamma_L = 1/2$. The remaining follows from Remark 1 (a).

If $0 < \alpha < 2$ we have by (15)

$$E|L(1)|^{1/\gamma} < \infty \quad \text{if} \quad \frac{1}{\alpha\gamma} < 1$$

and

$$E|L(1)|^{1/\gamma} = \infty \quad \text{if} \quad \frac{1}{\alpha\gamma} > 1.$$

It follows that $\lim_{t\to\infty} \varphi_\gamma(t) = 0$ if $\gamma > 1/\alpha$ and $\lim_{t\to\infty} \varphi_\gamma(t) = \infty$ if $\gamma < 1/\alpha$. From Proposition 1 we have $\gamma_L = 1/\alpha$. Since $\alpha < 2$ we have $E|L(t)|^2 = \infty$. Thus $\sigma_L^2(x,t) = D_L = \infty$. □

Next, we relate the diffusion index with regularly varying functions. We say that a non-negative function $\varphi(\cdot)$ is $\rho$-regularly varying at infinity if $\lim_{t\to\infty} \frac{\varphi(tx)}{\varphi(t)} = x^\rho$ for all $x > 0$ and we say that $\varphi(\cdot)$ is slowly varying at infinity if $\rho = 0$.

**Proposition 6.** For a Lévy stable process $L = L_{\alpha,\sigma}$ let $\varphi_t(x) = P(L(t) > x)$. Then,

$$\varphi_t(\cdot) \ \text{ is slowly varying at infinity if } \ \alpha = 2$$

$$\text{and} \quad \varphi_t(\cdot) \ \text{ is } \left(-\frac{1}{\gamma_L}\right) - \text{regularly varying at infinity if } \ 0 < \alpha < 2,$$

being $\gamma_L = \frac{1}{\alpha}$, the diffusion index.

**Proof.** Since $L(0) = 0$ we have $L(t) \stackrel{D}{=} S_{\alpha,\sigma t^{1/\alpha}}$. If $\alpha = 2$ we have $S_{2,\sigma\sqrt{t}} = N(0, 2\sigma^2 t)$, which has slowly varying tail. If $0 < \alpha < 2$ we make use of the following property of stable symmetric distribution : for $Z \stackrel{D}{=} S_{\alpha,\beta}$ we have

$$\lim_{z\to\infty} z^{\alpha} P(Z > z) = C_{\alpha}\frac{\beta^{\alpha}}{2}$$

(cf. [1]). If follows that

$$\lim_{z\to\infty} z^{\alpha} P(L(t) > z) = C_{\alpha}\frac{\sigma^{\alpha} t}{2}$$

and for $x > 0$ and $s > 0$ we have $\lim_{s\to\infty} \frac{\varphi_t(sx)}{\varphi_t(s)} = x^{-\alpha}$. By Proposition 5, $\gamma_L = 1/\alpha$. □

## 3. EXAMPLES

**Example 1.** Let $\{X(t) = X, t \geq 0\}$ where $X$ possesses zero-mean and finite moments. Then $\lim_{t\to\infty} \frac{E|X|^{1/\gamma}}{t} = 0$, $\forall\gamma > 0$. We have a subdiffusion regime with $\gamma_X = D_X = 0$.

**Example 2.** Let $\{B_{\sigma}(t) : t \geq 0\}$ be the Brownian motion with variance $\sigma^2/2$. Since $B_{\sigma}(\cdot)$ is the Lévy stable process $L_{1/2,\sigma}$ we have from Proposition 5 a normal diffusive regime with

$$\sigma^2_{B_{\sigma}}(x,t) = \sigma^2, \quad D_{B_{\sigma}} = \frac{\sigma^2}{2} \text{ and } \gamma_{B_{\sigma}} = \frac{1}{2}.$$

Now, consider the drifted Brownian motion $X(t) = B_{\sigma}(t) + \mu(t)$, $\mu \neq 0$ then $\sigma^2_X(x,t) = \sigma^2$. For the diffusion constant we have

$$D_X = \lim_{t\to\infty} \frac{E(X^2(t))}{2t} = \lim_{t\to\infty} \frac{\sigma^2 t + \mu^2 t^2}{2t} = \infty$$

which indicates a superdiffusion regime caused by the drift. But, if the centered process in considered we have

$$X(t) - E(X(t)) = X(t) - \mu t = B_{\sigma}(t)$$

and $\gamma_{B_{\sigma}} = 1/2$ (normal diffusion).

**Example 3.** Let $\{V(t) : t \geq 0\}$ be the Ornstein-Uhlenbeck given by Langevin equation

$$dV(t) = -\mu V(t)dt + \sigma dB(t)\ ,\ \mu > 0\ ,\ \sigma > 0$$

$$V(0) = 0.$$

Then $V(t) = \sigma \int_0^t e^{-\mu(t-x)} dB(s)$ which is a zero-mean Gaussian process with covariance function given by

$$\mathrm{cov}(V(t), V(s)) = \frac{\sigma^2}{2\mu}\left[e^{-\mu(t-s)} - e^{-\mu(t+s)}\right].$$

Also, for $h > 0$

$$E\left(V(t+h)|V(t)\right) = e^{-\mu h} V(t)$$

$$E(V^2(t+h)|V(t)) = e^{-2\mu h} V^2(t) + \frac{\sigma^2}{2\mu}(1 - e^{-2\mu h}).$$

It follows that

$$E\left(V(t+h) - V(t)|V(t)\right) = -\mu h V(t) + o(h)$$

and

$$E((V(t+h) - V(t))^2 \,|V(t)) = \sigma^2 h + o(h).$$

Where $o(h) \to 0$ as $h \to 0$. Hence, the infinitesimal coefficients are $\mu_V(x,t) = -\mu x$ and $\sigma_V^2(x,t) = \sigma^2$.

From the covariance function we have

$$D_V = \lim_{t\to\infty} \frac{E(V^2(t))}{2t} = \frac{\sigma^2}{2\mu} \lim_{t\to\infty} \frac{1}{2t}(1 - e^{-2\mu t}) = 0.$$

Alternatively, by a change of time and a rescaling of the state variable of the Brownian motion we can write

$$V(t) = e^{-\mu t} B\left(\frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)\right). \tag{16}$$

Using the self-similarity property of Brownian motion we have for $\gamma > 0$,

$$E|V(t)|^{1/\gamma} = e^{-\mu t/\gamma}\left[\frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)\right]^{1/2\gamma} E|B(1)|^{1/\gamma}$$

and $\lim_{t\to\infty} \frac{E|V(t)|^{1/\gamma}}{t} = 0, \ \forall \gamma > 0$. From (5) we have $\gamma_V = 0$ which indicates a sub-diffusion regime, as one would expect from representation (16). In fact, one can show that $V(t) \xrightarrow{D} N(0, \sigma^2/2\mu)$ and by Example 2 we have subdiffusion ($\xrightarrow{D}$: converge in distribution).

**Example 4.** Let $\{B^H(t) : t \geq 0\}$ be the fractional Brownian motion with Hurst parameter $0 < H < 1$, that is, a stochastic process satisfying :

(i) $B^H(0) = 0$ a.s.

(ii) $B^H(\cdot)$ is a zero-mean Gaussian process with covariance function given by

$$\operatorname{cov}(B^H(t), B^H(s)) = \frac{\sigma^2}{2}[t^{2H} + s^{2H} - |t-s|^{2H}] \tag{17}$$

where $\sigma^2 = \operatorname{var}(B^H(1))$.

For $H = 1/2$ we have the Brownian motion with variance $\sigma^2$ and for $0 < H < 1$ one can verify that $B^H(\cdot)$ is a self-similar process with similarity index $H$ and stationary increments (see, for example, [12]).

Since $(B^H(t+h), B^H(t))$ is a bivariate Gaussian with covariance function (17) we have

$$E(B^H(t+h)|B^H(t)) = \frac{(t+h)^{2H} + t^{2H} - h^{2H}}{2t^{2H}} B^H(t)$$

and

$$\operatorname{var}(B^H(t+h)|B^H(t)) = \frac{\sigma^2}{2} 2(t+h)^{2H} - \frac{[(t+h)^{2H} + t^{2H} - h^{2H}]^2}{2t^{2H}} \frac{\sigma^2}{2}.$$

It follows that

$$E(B^H(t+h) - B^H(t)|B^H(t) = x) = \frac{1}{2t^{2H}}[(t+h)^{2H} - t^{2H} - h^{2H}]x$$

and

$$\begin{aligned} E\{(B^H(t+h) - B^H(t))^2|B^H(t) = x\} &= \operatorname{var}(B^H(t+h)|B^H(t) = x) \\ &\quad + [E(B^H(t+h) - B^H(t)|B^H(t) = x)]^2. \end{aligned}$$

From (1) we have infinitesimal coefficients

$$\mu_{B^H}(x,t) = \begin{cases} -\infty & H < \frac{1}{2} \\ 0 & H = \frac{1}{2} \\ 2Ht^{2H-1}x & H > \frac{1}{2} \end{cases}$$

and

$$\sigma^2_{B^H}(x,t) = \begin{cases} \infty & H < \frac{1}{2} \\ \sigma^2 & H = \frac{1}{2} \\ 0 & H > \frac{1}{2}. \end{cases}$$

For the diffusion constant we use the self-similarity property $B^H(t) \stackrel{D}{=} t^H B^H(1)$. It follows that $E((B^H(1))^2) = \sigma^2,\ E((B^H(t))^2) = t^{2H}\sigma^2$ and

$$D_{B^H} = \lim_{t\to\infty} \frac{t^{2H}\sigma^2}{2t} = \begin{cases} 0 & H < \frac{1}{2} \\ \frac{\sigma^2}{2} & H = \frac{1}{2} \\ +\infty & H > \frac{1}{2}. \end{cases}$$

The diffusion index will provide a refinement of these results. For $\gamma > 0$ we have $E(|B^H(1)|^{1/\gamma}) < \infty$. Since $B^H(1)$ is Gaussian and

$$E(|B^H(t)|^{1/\gamma}) = E\{t^{H/\gamma}|B^H(1)|^{1/\gamma}\}.$$

Thus

$$0 < \lim_{t\to\infty} \frac{(|B^H(t)|^{1/\gamma})}{t} < \infty \ \text{ if and only } \ H = \gamma.$$

By Corollary 1 we have $\gamma_{B^H} = H$. Clearly, if $H = 1/2$ we have normal diffusion and $H < 1/2$ ( $H > 1/2$ ) represents subdiffusion (superdiffusion).

## 4. INFERENCE

As we have seen, the Lévy stable processes $L_{\alpha,\sigma}$ have diffusion index $\gamma_L \geq 1/2$ and Proposition 6 suggests that the tails of a stable law behave as $x^{-\alpha}$. For $0 < \alpha < 2$ this indicates a heavy-tail. On the other hand, extreme value distributions have been used to model heavy-tailed phenomena. The extreme value distributions (maximum) can be summarized by

$$\mathcal{G}_\beta(x) = \exp\{-(1+\beta x)^{-1/\beta}\}, \quad 1 + \beta x > 0,$$

where $\beta$ is the tail parameter and corresponding to $\beta > 0$ we have heavy-tailed distributions (Fréchet distributions).

Let $Y_1, Y_2, \cdots$ be i.i.d. random variables with a common distribution $F$. From the extreme value theory if $1 - F$ is $(-1/\beta)$-regularly varying at infinity for some $\beta > 0$ then there exist constants $c_n > 0$ and $d_n$ such that

$$P\left(\frac{\max\{Y_1, \cdots, Y_n\} - d_n}{c_n} \leq x\right) \xrightarrow{n} \mathcal{G}_\beta(x).$$

This suggests that the extreme order statistics have a role to play. And the approach for estimating the tail index $\beta$ leads to an estimate of the stability index $\alpha$ as well. We will make use of the classical Hills's estimator [6]: let $Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(n)}$ denote the order statistics of the sample $(Y_1, Y_2, \ldots, Y_n)$; let $k = k_n$ with $k_n/n \xrightarrow{n} 0$, then

$$\hat{\beta}_n = \frac{1}{k} \sum_{j=n-k}^{n} \log \frac{Y_{(j)}}{Y(n-k)} \xrightarrow{p} \beta$$

($\xrightarrow{p}$ : convergence in probability).

**Proposition 7.** For a Lévy stable process $L = L_{\alpha,\sigma}$ with $0 < \alpha < 2$ the diffusion index $\gamma_L$ can be estimated by

$$\frac{1}{k} \sum_{j=n-k}^{n} \log \frac{Y_{(j)}}{Y_{(n-k)}} \xrightarrow{p} \gamma_L \tag{18}$$

where $Y_j = L(j) - L(j-1)$ and $Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(n)}$ are the corresponding order statistics.

**Proof.** Since $L_{\alpha,\sigma}$ has stationary and independent increments, $L(0) = 0$ and $L(1) \stackrel{D}{=} S_{\alpha,\sigma}$ we have

$$L(1) - L(0), L(2) - L(1), \ldots, L(n) - L(n-1)$$

are i.i.d. random variables with a common distribution $S_{\alpha,\sigma}$. From Proposition 6 the tail of $S_{\alpha,\sigma}$ is $(-1/\gamma_L)$-regularly varying at infinity and (18) follows. □

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